

Positronium formation as a three-body reaction. II. The second-order nuclear amplitudes

F. Shojaei,¹ E. Ghanbari-Adivi,² M. J. Brunger,³ and M. A. Bolorizadeh^{1,a)}

¹*Physics Department, Shahid Bahonar University of Kerman, Kerman, 76169, Iran
and ICST, Mahan, 76315, Iran*

²*Physics Department and Isfahan Quantum Optics Group,
University of Isfahan, Isfahan, 81746, Iran*

³*Centre for Antimatter-Matter Studies, School of Chemistry, Physics & Earth Sciences,
Flinders University, Adelaide South Australia, Australia, 5042*

(Received 5 December 2007; accepted 17 October 2008; published online 5 January 2009)

We derive an exact analytic form for the second-order nuclear amplitudes, under the Faddeev three-body approach, which is applicable to the nonrelativistic high energy impact interaction where positronium is formed in the collision of a positron with an atom. © 2009 American Institute of Physics. [DOI: [10.1063/1.3032503](https://doi.org/10.1063/1.3032503)]

I. INTRODUCTION

There is continued interest in the interaction of positrons with atoms particularly when positronium Ps is formed due to the transfer of a target electron to the scattered positron.¹ Among the various possible processes that might occur in different collision systems, positronium formation by positron impact on atomic hydrogen is an example for one of the simplest quantal rearrangement processes that can occur in nature. Unfortunately, the theoretical investigation of this simple rearrangement process involves the full complexities of the quantum mechanical three-body problem and as such is too difficult to solve exactly.² These complexities thus compel physicists to use different theories for different ranges of the impact energy, with one such approach being given here. Note that, in general, the interaction is a three-body interaction for the simple hydrogen target and a many-body interaction for larger target atoms.

An investigation was therefore initiated to study positronium formation in the interaction of an energetic positron, with atomic hydrogen as the target, under a three-body approach. Note that the long-range Coulomb interaction plays an important role in this three-body interaction. A fully quantum mechanical three-body approach, which is particularly applicable at moderate and high impact energies for charge transfer interactions, is the Faddeev–Watson–Lovelace (FWL) theory.^{3–9} In this formalism, the charge transfer amplitude is evaluated by solving a set of three coupled integral equations in which the two-body T -matrices play the main role. A formal iteration series is obtained incorporating the Faddeev equations, and it should be noted that the validity of the Faddeev equations for Coulomb interactions is not proven here mathematically. However, our method is successful when compared with the available experimental results in the study of charge transfer¹⁰ and ionization¹¹ of atoms by projectile ions where the Coulomb interaction is involved. A recent study¹² of the charge transfer process in p -He collisions shows that both the Faddeev method, and the continuum distorted-wave (CDW) method need improvement in order to describe the experimental results. Note that we will refer the reader to Ref. 13, which we hereafter called Paper I, for a more detailed discussion of the pros and cons of the current method.

The experimental data on positronium formation are limited. Sperber *et al.*^{14,15} measured the total cross sections for Ps production in e^+ -H collisions for the energy range of 13–55 eV. Calcu-

^{a)} Author to whom correspondence should be addressed. Electronic mail: mabolori@mail.uk.ac.ir.

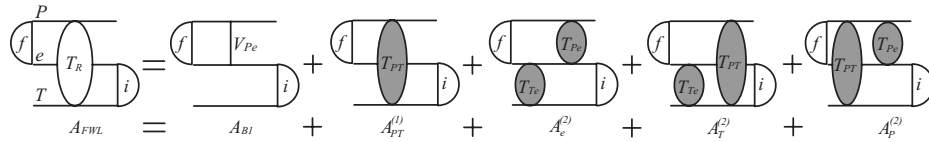


FIG. 1. The perturbation diagram associated with the five terms of the FWL amplitude for charge exchange.

lations were performed for a similar process up to 50 000 eV impact energy, incorporating the impulse approximation (IA),¹⁶ strong potential Born (SPB) approximation,¹⁷ and second Born approximation.¹⁸

Here we consider positronium formation in the collision of positrons with atomic hydrogen as being a three-body reaction, where we find an exact form for the second-order nuclear interaction amplitudes. The present method is applicable to charge exchange at high but nonrelativistic energies, where the speed of the projectile is higher than the speed of electrons in the target by one to two orders of magnitude. As noted earlier this paper is a continuation of our previously published work,¹³ Paper I. As was discussed in detail in Paper I, the interaction amplitude, under the FWL approach and up to second order, assuming the state of the initial channel is $|i\rangle$ and the final channel is $|f\rangle$, can be written as

$$A_{\text{FWL}} = \underbrace{\langle f|V_{Pe}|i\rangle}_{A_{B1}} + \underbrace{\langle f|T_{PT}|i\rangle}_{A_{PT}^{(1)}} + \underbrace{\langle f|T_{Te}G_0^{(+)}T_{Pe}|i\rangle}_{A_e^{(2)}} + \underbrace{\langle f|T_{Te}G_0^{(+)}T_{PT}|i\rangle}_{A_T^{(2)}} + \underbrace{\langle f|T_{PT}G_0^{(+)}T_{Pe}|i\rangle}_{A_P^{(2)}}, \quad (1)$$

where V_{xy} , T_{xy} with subscripts are the respective two-body interaction potentials and the two-body transition matrices and $G_0^{(+)}$ is the free Green's function operator. The second-order interaction amplitude therefore includes five terms which are shown schematically in Fig. 1.

The first two amplitudes of Eq. (1) are to first order, while the remaining three amplitudes are the second-order terms. In Paper I, the first of these second-order amplitudes, responsible for the presence of the Thomas peak in the angular distribution of the positronium formation differential cross section, was calculated and discussed in detail. Hence here we concentrate on the second ($A_T^{(2)}$) and third ($A_P^{(2)}$) second-order terms of Eq. (1), which are the terms responsible for the generalized Thomas peak. Note that we refer to the terms $A_T^{(2)}$ and $A_P^{(2)}$ as the second-order internuclear term associated with the target and projectile, respectively, and we derive a closed form for them in this paper. Further note that in the charge transfer process due to the positron-hydrogen collision, both the Thomas peak and generalized Thomas peak coincide at 45° in the angular distribution of the cross sections.

The notations defined throughout Paper I will also be used here. However, for completeness, the commonly referred to terms will again be defined here. The structure of the present paper is as follows. The second-order internuclear terms will be introduced and simplified in Sec. II. In that same section, a closed form for the nuclear interaction amplitudes is subsequently derived. This new form enables us to calculate, in a manner described in Sec. III, the second-order nuclear amplitudes quite simply and efficiently with a personal computer, thereby saving dramatically on the computational cost of the calculations. Concluding remarks and a discussion of the present framework are finally given in Sec. IV. Note that atomic units are used throughout this manuscript.

II. THE SECOND-ORDER NUCLEAR AMPLITUDES

Let us consider a three-body process in which a projectile P of effective charge Z_P impacts on a target subsystem consisting of an electron e bound to a positive target ion T of effective charge Z_T and mass M_T . The rearrangement scattering process is denoted as

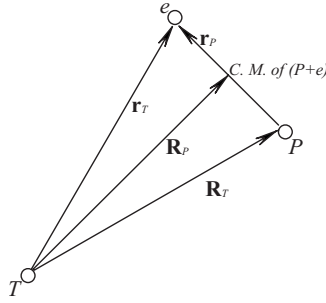


FIG. 2. The Jacobi coordinates when the mass of the target's electron and the projectile (positron) are negligibly smaller than those of the target's nucleus.



in which the parentheses denote the bound electron-ion aggregates. The Faddeev–Watson multiple-scattering series for the transition matrix \mathcal{T}_R , describing the charge transfer process, when truncated to second order, is⁷⁻⁹

$$\mathcal{T}_R^{(2)} = V_{Pe} + T_{PT} + T_{Te}G_0^+T_{Pe} + T_{Te}G_0^+T_{PT} + T_{PT}G_0^+T_{Pe}. \quad (3a)$$

The second-order FWL amplitude A_{FWL} for the charge transfer process is, therefore, given by

$$A_{\text{FWL}} = \langle f | \mathcal{T}_R^{(2)} | i \rangle = A_e + A_n, \quad (3b)$$

in which A_e and A_n are the nuclear-electronic and the internuclear components of the amplitude, respectively. Note that $|i\rangle$ and $|f\rangle$ are the states which describe the system in the initial and final channels. The internuclear component of Eq. (3b) includes a first-order term $A_{PT}^{(1)}$ and two second-order terms $A_T^{(2)}$ and $A_P^{(2)}$,

$$A_n = \langle f | T_{PT} | i \rangle + \langle f | T_{Te}G_0^+T_{PT} | i \rangle + \langle f | T_{PT}G_0^+T_{Pe} | i \rangle = A_{PT}^{(1)} + A_T^{(2)} + A_P^{(2)}. \quad (4)$$

The projectile's mass M_p is equal to the target's electron mass m in our application and both these masses are small when compared with that of the target mass $m = M_p \ll M_T$. Accordingly, the center of mass of the whole three-body system approximately coincides with the target T . It is also assumed in our analysis that the target ion remains at rest during the collision. The remaining six degrees of freedom for the constituents of the system are now described by the Jacobi coordinate vectors $(\mathbf{r}_T, \mathbf{R}_T)$ and $(\mathbf{r}_P, \mathbf{R}_P)$, as shown in Fig. 2.

Let us define the quantities \mathbf{K}_i , \mathbf{K}_f , ε_{n_i} , and ε_{n_f} as the momentum of the incident particle, the momentum of the final bound product, the initial bound-state energy, and the final bound-state energy, respectively. It therefore follows that the conservation of the total collision energy is given by

$$E = K_i^2/2 + \varepsilon_{n_i} = K_f^2/4 + \varepsilon_{n_f} = \mathbf{v}_i^2/2 + \varepsilon_{n_i} = \mathbf{v}_f^2 + \varepsilon_{n_f}, \quad (5a)$$

in which \mathbf{v}_i and \mathbf{v}_f are magnitudes of the initial and final velocities: $\mathbf{v}_i = \mathbf{K}_i$ and $\mathbf{v}_f = \mathbf{K}_f/2$, respectively. The momentum transfers experienced by the target and the projectile during the collision have been defined as

$$\mathbf{J} = \mathbf{K}_i - \mathbf{K}_f = \mathbf{v}_i - 2\mathbf{v}_f \quad (5b)$$

and

$$\mathbf{K} = \frac{1}{2}\mathbf{K}_f - \mathbf{K}_i = \mathbf{v}_f - \mathbf{v}_i, \quad (5c)$$

respectively. Finally, momentum conservation for the charge transfer process reads as $\mathbf{K} + \mathbf{J} + \mathbf{v}_f = 0$ or $2\mathbf{K} + \mathbf{J} + \mathbf{v}_i = 0$, where the projectile is specified as a positron. The method described in Sec. III of Paper I can now be used to expand the plan-wave phases of the initial and final wave functions as

$$\mathbf{k}_i \cdot \mathbf{r}_T + \mathbf{K}_i \cdot \mathbf{R}_T = \frac{1}{2}(\mathbf{k}_i - \mathbf{K}_i) \cdot \mathbf{r}_p + (\mathbf{k}_i + \mathbf{K}_i) \cdot \mathbf{R}_p, \quad (6a)$$

$$\mathbf{k}_f \cdot \mathbf{r}_p + \mathbf{K}_f \cdot \mathbf{R}_p = (\mathbf{k}_f + \frac{1}{2}\mathbf{K}_f) \cdot \mathbf{r}_T + (-\mathbf{k}_f + \frac{1}{2}\mathbf{K}_f) \cdot \mathbf{R}_T, \quad (6b)$$

where \mathbf{k}_i and \mathbf{k}_f are the electron momenta in the initial and final bound states. Substituting the initial and final wave functions for the initial and final bound-state subsystems into the integral form of the fourth term of Eq. (1), one gets

$$A_T^{(2)} = (2\pi)^{-3} \int d\mathbf{k}_f d\mathbf{k}_i \phi_f^*(\mathbf{k}_f) \phi_i(\mathbf{k}_i) \langle \mathbf{k}_f, \mathbf{K}_f | T_{Te} G_0^{(+)} T_{PT} | \mathbf{k}_i, \mathbf{K}_i \rangle, \quad (7)$$

where $\phi_i(\mathbf{k}_i)$ and $\phi_f(\mathbf{k}_f)$ are the initial and final wave functions of the active electron in momentum space. The inner integral is now denoted as

$$M_1 = \langle \mathbf{k}_f, \mathbf{K}_f | T_{Te} G_0^{(+)} T_{PT} | \mathbf{k}_i, \mathbf{K}_i \rangle = (2\pi)^{-6} \int d\mathbf{k}_1 d\mathbf{K}_1 \langle \mathbf{k}_f, \mathbf{K}_f | T_{Te} | \mathbf{k}_1, \mathbf{K}_1 \rangle \langle \mathbf{k}_1, \mathbf{K}_1 | G_0^{(+)} T_{PT} | \mathbf{k}_i, \mathbf{K}_i \rangle, \quad (8a)$$

where \mathbf{k}_1 and \mathbf{K}_1 are the three-dimensional dummy variables. Changing from the Jacobi coordinates \mathbf{r}_T and \mathbf{R}_T to \mathbf{r}_p and \mathbf{R}_p , as shown by Eqs. (6a) and (6b), the integral form of M_1 changes to

$$M_1 = \langle \mathbf{k}_f, \mathbf{K}_f | T_{Te} G_0^{(+)} T_{PT} | \mathbf{k}_i, \mathbf{K}_i \rangle = (2\pi)^{-6} \int d\mathbf{k}_1 d\mathbf{K}_1 \langle \mathbf{k}_f + \mathbf{K}_f/2, -\mathbf{k}_f + \mathbf{K}_f/2 | T_{Te} | \mathbf{k}_1, \mathbf{K}_1 \rangle \\ \times \langle \mathbf{K}_1, \mathbf{k}_1 | G_0^{(+)} T_{PT} | \mathbf{K}_i, \mathbf{k}_i \rangle. \quad (8b)$$

The above integral form of M_1 can be further simplified to

$$M_1 = \int d\mathbf{k}_1 d\mathbf{K}_1 T_{Te}(\mathbf{k}_f + \mathbf{K}_f/2, \mathbf{k}_1; E_f) G_0^{(+)}(E_n) T_{PT}(\mathbf{K}_1, \mathbf{K}_i; E_n) \delta(-\mathbf{k}_f + \mathbf{K}_f - \mathbf{K}_1) \delta(\mathbf{k}_1 - \mathbf{k}_i) \quad (8c)$$

by implementing some simple algebraic rearrangement of the terms. Integration over the momentum vectors \mathbf{k}_1 and \mathbf{K}_1 will easily simplify M_1 further to give

$$M_1 = T_{Te}(\mathbf{k}_f + \mathbf{K}_f/2, \mathbf{k}_i; E_f) G_0^{(+)}(E_n) T_{PT}(-\mathbf{k}_f + \mathbf{K}_f, \mathbf{K}_i; E_n). \quad (8d)$$

This final result for M_1 is now substituted into Eq. (7) to obtain the general integral form of the second-order nuclear partial amplitude associated with the target as

$$A_T^{(2)} = (2\pi)^{-3} \int d\mathbf{k}_f d\mathbf{k}_i \phi_f^*(\mathbf{k}_f) T_{Te}(\mathbf{k}_f + \mathbf{v}_f, \mathbf{k}_i; E_f) G_0^{(+)}(E_n) T_{PT}(\mathbf{v}_f - \mathbf{k}_f, \mathbf{v}_i; E_n) \phi_i(\mathbf{k}_i). \quad (7')$$

The scattering energies E_f and E_n are, respectively, given by

$$E_f = E - \frac{1}{2}(\mathbf{k}_f - \frac{1}{2}\mathbf{K}_f)^2$$

and

$$E_n = E - \frac{1}{2}k_i^2 = E - \frac{1}{2}(\mathbf{k}_f + \frac{1}{2}\mathbf{K}_f)^2.$$

The second-order nuclear partial amplitude associated with the projectile, the fifth term in Eq. (1), is next simplified using the same procedure as that in arriving at Eq. (7). In doing this the second-order nuclear amplitude associated with the projectile becomes

$$A_P^{(2)} = (2\pi)^{-3} \int d\mathbf{k}_f d\mathbf{k}_i \phi_f^*(\mathbf{k}_f) \phi_i(\mathbf{k}_i) \langle \mathbf{k}_f, \mathbf{K}_f | T_{PT} G_0^{(+)} T_{Pe} | \mathbf{k}_i, \mathbf{K}_i \rangle. \quad (9)$$

The inner factor in Eq. (9), denoted as M_2 , is rewritten and simplified as

$$\begin{aligned} M_2 = \langle \mathbf{k}_f, \mathbf{K}_f | T_{PT} G_0^{(+)} T_{Pe} | \mathbf{k}_i, \mathbf{K}_i \rangle &= (2\pi)^{-6} \int d\mathbf{k}_2 d\mathbf{K}_2 \langle -\mathbf{k}_f + \mathbf{K}_f/2, \mathbf{k}_f + \mathbf{K}_f/2 | T_{PT} | \mathbf{k}_2, \mathbf{K}_2 \rangle \\ &\times \langle \mathbf{K}_2, \mathbf{k}_2 | G_0^{(+)} T_{Pe} | (\mathbf{k}_i - \mathbf{K}_i)/2, \mathbf{k}_i + \mathbf{K}_i \rangle, \end{aligned} \quad (10a)$$

where \mathbf{k}_2 and \mathbf{K}_2 are the approximate three-dimensional dummy variables. The bra-ket terms in Eq. (10a) can further be simplified by the same procedure described in arriving at Eq. (8c), so that

$$\begin{aligned} M_2 &= \int d\mathbf{k}_2 d\mathbf{K}_2 T_{PT}(\mathbf{k}_f + \mathbf{K}_f/2, \mathbf{k}_2; E_f) G_0^{(+)}(E_n) T_{Pe}(\mathbf{k}_2, (\mathbf{k}_i - \mathbf{K}_i)/2; E_n) \\ &\times \delta(-\mathbf{k}_f + \mathbf{K}_f/2 - \mathbf{k}_2 - \mathbf{K}_2/2) \delta(\mathbf{K}_2 - \mathbf{k}_i - \mathbf{K}_i). \end{aligned} \quad (10b)$$

This penultimate form of M_2 is integrated over \mathbf{k}_2 and \mathbf{K}_2 , giving its final form as

$$M_2 = T_{PT}(-\mathbf{k}_f + \mathbf{K}_f/2, -\mathbf{k}_f + \mathbf{K}_f/2 + \mathbf{k}_i + \mathbf{J}; E_n) G_0^{(+)}(E_i) T_{Pe}(\mathbf{k}_f - (\mathbf{k}_i + \mathbf{J})/2, (\mathbf{k}_i + \mathbf{K}_i)/2; E_i). \quad (10c)$$

The resulting second-order nuclear amplitude associated with the projectile thus takes the form

$$\begin{aligned} A_P^{(2)} &= (2\pi)^{-3} \int d\mathbf{k}_f d\mathbf{k}_i \phi_f^*(\mathbf{k}_f) \phi_i(\mathbf{k}_i) T_{PT}(\mathbf{v}_f - \mathbf{k}_f, \mathbf{k}_i - \mathbf{k}_f - \mathbf{K}; E_n) \\ &\times G_0^{(+)}(E_i) T_{Pe}(\mathbf{k}_f - \mathbf{k}_i/2 - \mathbf{J}/2, (\mathbf{k}_i + \mathbf{v}_i)/2; E_i), \end{aligned} \quad (9')$$

where the scattering energy E_i is defined as

$$E_i = E - \frac{1}{4}(\mathbf{k}_i + \mathbf{K}_i)^2.$$

III. EVALUATION OF THE AMPLITUDE

The FWL amplitude, its five partial components being summed in Eq. (1), describes the transfer of the active electron to the projectile to form positronium. In practice we evaluate this amplitude by representing it with an initial wave packet centered about the target ion, which scatters through multiple collisions to form a final wave packet centered about the receding projectile ion. As noted in Sec. I, the scattering in each of the two-body collisions is expressed in terms of a two-body transition operator and a free Green's function.¹⁹ The second-order nuclear partial amplitudes, therein the second-order Faddeev amplitude, as given by Eqs. (7) and (9), are now evaluated in this section.

The first second-order internuclear term $A_T^{(2)}$ describes a double-scattering mechanism involving two subsequent scatterings of the positron and the electron by the target ion, the next result being that the positron suffers a collision with the target nucleus as well as the active electron to form a positronium subsystem in the final channel. Beginning from the far right-hand side of Eq. (7), the electronic wave function of the bound electron with momentum distribution \mathbf{k}_i in the initial channel is given by $\phi_i(\mathbf{k}_i)$. In the projectile frame, this wave packet is seen as an electronic cloud with its momentum distribution centered at $(\mathbf{k}_i + \mathbf{v}_i)/2$ for which the energy corresponding to

\mathbf{k}_i is \mathbf{E}_i . The wave packet colliding with the projectile suffers a transfer of momentum $\mathbf{k}_f - \mathbf{v}_i$, such that after the collision the momentum becomes $\mathbf{k}_f + (\mathbf{k}_i + \mathbf{J})/2$. The free propagation of this wave packet to the next collision is represented by the (completely) free Green's function. Switching now to the target frame, one must boost the momentum of the wave packet by $(\mathbf{k}_i + \mathbf{v}_i)/2$, so that its final momentum becomes $\mathbf{k}_i + \mathbf{k}_f - \mathbf{K}$. The corresponding shift in energy is given by $\Delta E = (\mathbf{k}_i + \mathbf{k}_f - \mathbf{K})^2/2 - (\mathbf{k}_i/2 + \mathbf{k}_f + \mathbf{J}/2)^2/2$. Thus, in the target frame, the wave-packet components have the energies $E_f = E_i + \Delta E$. The second collision occurs with the target ion and causes a momentum transfer $-\mathbf{k}_i - \mathbf{J}$, giving a final momentum of $\mathbf{k}_f + \mathbf{v}_f$. The transition matrix T_{Te} thus describes the off-energy-shell scattering of the wave packet in the second collision. After this second collision the wave packet has a distribution of momentum \mathbf{k}_f centered approximately about \mathbf{v}_f . This distribution is given by $\phi_f^*(\mathbf{k}_f)$ in the projectile frame.

In order to evaluate the second-order nuclear partial amplitudes, let us assume that the velocity of the projectile is large compared with the velocity of the active electron in the initial and/or the final bound states. Under this assumption, Alston²⁰ derived a simplified form for the two-body Coulomb transition operators. Following this prescription we let \mathbf{k} , \mathbf{k}' , and E be the initial channel momentum, the final channel momentum, and the channel energy of a two-body interaction, respectively. For the case of a nonrelativistic high energy projectile, the Coulomb transition matrix has been simplified previously,²¹

$$T_{xy}(\mathbf{k}, \mathbf{k}'; E) = 4\pi Z_x Z_y e^{-\pi\nu_{xy}^a} \frac{\Gamma(1 - i\nu_{xy}^a) \Gamma(1 + i\nu_{xy})}{\Gamma(1 - i\nu_{xy})} (2\mu_{xy}E - k^2)^{i\nu_{xy}^a} (2\mu_{xy}E - k'^2)^{i\nu_{xy}^a} (8\mu_{xy}E)^{i(\nu_{xy} - 2\nu_{xy}^a)} \times |\mathbf{k} - \mathbf{k}'|^{-2-2i\nu_{xy}}, \quad (11)$$

where all the relevant parameters are defined in Paper I. This form of the transition matrix contains the off-shell factors and the Coulomb interaction amplitude. All the appropriate Sommerfeld parameters which appear, hereafter, are defined as $\nu_{Pe} = -Z_P/v_i$, $\nu_{Pe}^a = -Z_P^a/v_i$, $\nu_{Te} = -Z_T/v_f$, $\nu_{Te}^a = -Z_T^a/v_f$, $\nu_{PT} = Z_P Z_T/v_i$, and $\nu_{PT}^a = Z_P^a Z_T^a/v_i$, in which Z and Z^a are the projectile and target asymptotic charges, respectively.

The initial and final wave functions are labeled with two sets of quantum numbers (n_i, l_i, m_i) and (n_f, l_f, m_f) , and the normalization factors of the initial and final wave functions are denoted by N_{n_i, l_i} and N_{n_f, l_f} , respectively. The wave functions for the initial and final bound systems, in the momentum representation, are

$$\tilde{\phi}_{\text{state}}(\mathbf{k}) \equiv \tilde{\phi}_{nlm}(\mathbf{k}) = R_{nl}(k) Y_{lm}(\hat{\mathbf{k}}), \quad (12a)$$

where $R_{nl}(k)(Y_{lm}(\hat{\mathbf{k}}))$ is the usual radial (angular) component. Note that the indices i and f , designating the initial and final bound states, are eliminated for simplicity in this portion of the manuscript. The radial part of the wave function is now given by

$$R_{nl}(k) = N_{nl} \frac{k^l}{(k^2 + \alpha_n^2)^{l+2}} C_{n-l-1}^{l+1} \left(1 - \frac{2\alpha_n^2}{k^2 + \alpha_n^2} \right), \quad (12b)$$

where

$$N_{nl} = (2\pi)^{3/2} l! (\alpha_n)^{(2l+5)/2} ([2^{4l+5} n(n-l-1)!] [\pi(n+l)!])^{1/2},$$

$\alpha_n = \sqrt{-2\mu\epsilon_n} = \mu Z/n$ and $C_{n-l-1}^{l+1}(x)$ are the appropriate normalization factor, the positive parameter related to the square root of the electron energy in the bound state, and the Gegenbauer polynomial, respectively. In addition, μ is the reduced mass of electron in the bound system which has values of 1 for atomic hydrogen and 1/2 for positronium. Equation (12b) is further simplified by a unique expansion of the Gegenbauer polynomials in terms of powers of $(k^2 + \alpha_n^2)^{-1}$,

$$C_{n-l-1}^{l+1} \left(1 - \frac{2\alpha_n^2}{k^2 + \alpha_n^2} \right) = \sum_{s=0}^{n-l-1} A_{nl,s} (k^2 + \alpha_n^2)^{-s}, \quad (12c)$$

in which $A_{nl,s}$ is the expansion coefficient. Substituting Eq. (12c) into Eq. (12b), a convenient expanded form for $R_{nl}(k)$ is derived as

$$R_{nl}(k) = N_{nl} \sum_{s=0}^{n-l-1} A_{nl,s} k^l (k^2 + \alpha_n^2)^{-s-l-2}. \quad (12d)$$

Further substituting the appropriate parameters as defined by Eq. (4), into the free Green's function, leads to

$$[G_0^{(+)}(E_n)]^{-1} = [G_0^{(+)}(E_i)]^{-1} = (E_n - (\mathbf{v}_f - \mathbf{k}_f)^2/2 + i\eta), \quad \eta \rightarrow 0^+, \quad (13a)$$

and

$$[G_0^{(+)}(E_f)]^{-1} = [G_0^{(+)}(E_n)]^{-1} = (E_f - k_i^2/2 + \eta), \quad \eta \rightarrow 0^+. \quad (13b)$$

Now using the results of Eqs. (13a) and (13b) in the forms of T_{Te} and T_{TP} of Eq. (7'), one gets the general functions,

$$\begin{aligned} T_{Te}(\mathbf{k}_f + \mathbf{v}_f, \mathbf{k}_i; E_f) &= -4\pi Z_T e^{-\pi\nu_{Te}^a} \frac{\Gamma(1 - i\nu_{Te}^a)^2 \Gamma(1 + i\nu_{Te}^a)}{\Gamma(1 - i\nu_{Te}^a)} \\ &\quad \times (8E_f)^{i\nu_{Te}^a - 2i\nu_{Te}^a} [2E_f - (\mathbf{k}_f + \mathbf{v}_f)^2]^{i\nu_{Te}^a} \\ &\quad \times (2E_f - k_i^2)^{i\nu_{Te}^a} |\mathbf{k}_f + \mathbf{v}_f - \mathbf{k}_i|^{-2-2i\nu_{Te}^a} \end{aligned} \quad (13c)$$

and

$$\begin{aligned} T_{PT}(\mathbf{v}_f - \mathbf{k}_f, \mathbf{v}_i; E_n) &= 4\pi Z_P Z_T e^{-\pi\nu_{PT}^a} \frac{\Gamma(1 - i\nu_{PT}^a)^2 \Gamma(1 + i\nu_{PT}^a)}{\Gamma(1 - i\nu_{PT}^a)} \\ &\quad \times (8E_n)^{i\nu_{PT}^a - 2i\nu_{PT}^a} [2E_n - (\mathbf{v}_f - \mathbf{k}_f)^2]^{i\nu_{PT}^a} \\ &\quad \times (2E_n - v_i^2)^{i\nu_{PT}^a} |\mathbf{v}_f - \mathbf{k}_f - \mathbf{v}_i|^{-2-2i\nu_{PT}^a}. \end{aligned} \quad (13d)$$

The terms of the forms $2E_f - (\mathbf{k}_f + \mathbf{v}_f)^2$, $2E_n - (\mathbf{v}_f - \mathbf{k}_f)^2$, $2E_n - v_i^2$, $2E_n$, etc., are correspondingly simplified to $-2(k_f^2 - \varepsilon_f)$, $-2[(k_f^2 + v_f^2)/2 - \varepsilon_f]$, $-2[k_i^2 - \varepsilon_i/2]$, $v_i^2 + 2\varepsilon_i - k_i^2$, etc., respectively. Under the approximation that the speed of the active electron is significantly smaller than that of the projectile speed, the form of $A_T^{(2)}$ reduces to

$$\begin{aligned} A_T^{(2)} &= C_1 \int_0^\infty dk_i k_i^{l_i+2} (k_i^2 + \alpha_{n_i}^2)^{-l_i-2+i\nu_{PT}^a} C_{n_i-l_i-1}^{l_i+1} \left(\frac{k_i^2 - \alpha_{n_i}^2}{k_i^2 + \alpha_{n_i}^2} \right) \\ &\quad \times \int_0^\infty dk_f k_f^{l_f+2} (k_f^2 + \alpha_{n_f}^2)^{-l_f-2+i\nu_{Te}^a} C_{n_f-l_f-1}^{l_f+1} \left(\frac{k_f^2 - \alpha_{n_f}^2}{k_f^2 + \alpha_{n_f}^2} \right), \end{aligned} \quad (14a)$$

where

$$C_1 = 4Z_P Z_T^2 2^{i[-3(\nu_{Te}^a + \nu_{PT}^a) + 2(\nu_{Te} - \nu_{PT})]} N_{n_i l_i} N_{n_f l_f} \exp[-\pi(2\nu_{Te}^a + \nu_{PT}^a)] \delta_{l_0} \delta_{m_0} \delta_{l'_0} \delta_{m'_0} \\ \times \frac{\Gamma(1 - i\nu_{Te}^a)^2 \Gamma(1 + i\nu_{Te}) \Gamma(1 - i\nu_{PT}^a)^2 \Gamma(1 + i\nu_{PT})}{\Gamma(1 - i\nu_{Te}) \Gamma(1 - i\nu_{PT})} v_i^{-2-2i(\nu_{Te}^a + \nu_{PT}^a - \nu_{PT})} K^{-2+2i\nu_{PT}}. \quad (14b)$$

Substituting the special expansion of Eq. (12c) into Eq. (14a), the penultimate form for $A_T^{(2)}$ is deduced,

$$A_T^{(2)} = C_1 \sum_{s_i=0}^{n_i-l_i-1} \sum_{s_f=0}^{n_f-l_f-1} A_{n_f l_f s_f} A_{n_i l_i s_i} \int_0^\infty dk_i k_i^{l_i+2} (k_i^2 + \alpha_{n_i}^2)^{-s_i-l_i-2+i\nu_{PT}^a} \int_0^\infty dk_f k_f^{l_f+2} (k_f^2 + \alpha_{n_f}^2)^{-s_f-l_f-2+i\nu_{PT}^a}. \quad (14c)$$

The integrals contained within Eq. (14c) have a simple form, which can be calculated easily²² as

$$\int_0^\infty dk k^{l+2} (k^2 + \alpha^2)^{-s-l-2+i\nu} = \frac{\Gamma[(l+1+2s-2i\nu)/2] \Gamma[(l+3)/2]}{2\Gamma(l+s+2-i\nu) (\alpha)^{l+2s+1-2i\nu}}. \quad (14d)$$

This result was explained in detail in Paper I. Thus the result of our derivation is

$$A_T^{(2)} = C_1 \sum_{s_i=0}^{n_i-l_i-1} \sum_{s_f=0}^{n_f-l_f-1} A_{n_i l_i s_i} \frac{\Gamma[(l_i+3)/2]}{2\Gamma(l_i+s_i+2-i\nu_{PT}^a)} \frac{\Gamma[(l_i+1+2s_i-2i\nu_{PT}^a)/2]}{(-2\varepsilon_i)^{(l_i+1+2s_i-2i\nu_{PT}^a)/2}} \\ \times A_{n_f l_f s_f} \frac{\Gamma[(l_f+3)/2]}{2\Gamma(l_f+s_f+2-i\nu_{Te}^a)} \frac{\Gamma[(l_f+1+2s_f-2i\nu_{Te}^a)/2]}{(-\varepsilon_f)^{(l_f+1+2s_f-2i\nu_{Te}^a)/2}}. \quad (15)$$

Finally let us consider the next second-order nuclear amplitude $A_P^{(2)}$, which is the third double-scattering mechanism in the FWL approach. This term describes the Thomas mechanism in scattering of the final bound subsystem at the critical angle of 45° . In the first step, a positron knocks the active electron into its final state and in the second step a collision occurs between the positron and the target ion. After the second collision, the positron attains a final velocity of the order of the scattered electron and travels in almost the same direction. This therefore enables them to form positronium. $A_P^{(2)}$ is simplified, as from that given in Eq. (9'), by simplifying the terms T_{PT} , T_{Pe} , $G_0^{(+)}(E_i)$, and the initial and final wave functions of the bound systems. This is achieved by following the approach outlined previously for $A_T^{(2)}$, so that

$$T_{PT}(\mathbf{v}_f - \mathbf{k}_f, \mathbf{k}_i - \mathbf{k}_f - \mathbf{K}; E_n) = 4\pi Z_P Z_T e^{-\pi\nu_{PT}^a} \frac{\Gamma(1 - i\nu_{PT}^a)^2 \Gamma(1 + i\nu_{PT})}{\Gamma(1 + i\nu_{PT})} \\ \times (2E_n - (\mathbf{v}_f - \mathbf{k}_f)^2)^{i\nu_{PT}^a} \times (2E_n - (\mathbf{k}_i - \mathbf{k}_f - \mathbf{v}_f + \mathbf{v}_i)^2)^{i\nu_{PT}^a} \\ \times (8E_n)^{i\nu_{PT}-2i\nu_{PT}^a} |\mathbf{k}_i + \mathbf{J}|^{-2-2i\nu_{PT}} \quad (16a)$$

and

$$T_{Pe}(\mathbf{k}_f - \mathbf{k}_i/2 - \mathbf{J}/2, (\mathbf{k}_i + \mathbf{v}_i)/2; E_i) = -4\pi Z_P e^{-\pi\nu_{Pe}^a} \frac{\Gamma(1 - i\nu_{Pe}^a)^2 \Gamma(1 + i\nu_{Pe})}{\Gamma(1 - i\nu_{Pe})} \\ \times [2E_i - (\mathbf{k}_f - \mathbf{k}_i/2 - \mathbf{J}/2)^2]^{i\nu_{Pe}^a} [2E_i - (\mathbf{k}_i + \mathbf{v}_i)^2/4]^{i\nu_{Pe}^a} \\ \times (8E_i)^{i(\nu_{Pe}+2\nu_{Pe}^a)} |\mathbf{k}_f - \mathbf{k}_i + \mathbf{K}|^{-2-2i\nu_{Pe}}. \quad (16b)$$

The free Green's function in Eq. (9') is given by

$$G_0^{(+)}(E_i) = (E_i - (\mathbf{k}_i + \mathbf{v}_i)^2/2 + i\eta)^{-1}, \quad (16c)$$

where again the terms such as $E_n - (\mathbf{k}_i - \mathbf{k}_f - \mathbf{v}_f + \mathbf{v}_i)^2/2$, $E_i - |\mathbf{k}_f + \mathbf{k}_i/2 + \mathbf{J}/2|^2$, $E_n - (\mathbf{v}_f - \mathbf{k}_f)^2/2$, $E_i - (\mathbf{k}_i - \mathbf{v}_i)^2/4$, etc., are equal to $[G_0^+(E_n)]^{-1}$, $[G_0^{(+)}(E_i)]^{-1}$, $-(\mathbf{k}_f^2 - \varepsilon_{n_f})$, $-2(\mathbf{k}_i^2 - \varepsilon_{n_i})$, etc., respectively. We can, also, simplify the free Green's function as

$$\begin{aligned} G_0^{(+)}(E_f) &= (E_f - |\mathbf{k}_i + \mathbf{k}_f - \mathbf{K}|^2/2 + i\eta)^{-1} \\ &= \left[\frac{1}{2}v_f^2 - \frac{1}{2}k_f^2 - \mathbf{k}_f \cdot \mathbf{v}_f + \varepsilon_f - \frac{1}{2}(\mathbf{k}_i + \mathbf{k}_f)^2 - \frac{1}{2}K^2 + \mathbf{K} \cdot (\mathbf{k}_i + \mathbf{k}_f) + i\eta \right]^{-1} \\ &= \left[\frac{1}{2}(v_f^2 - K^2 + 2\varepsilon_f) + \mathbf{k}_i \cdot \mathbf{K} - \mathbf{k}_f \cdot \mathbf{J} + i\eta \right]^{-1} \\ &= [Ka + \mathbf{k}_i \cdot \mathbf{K} - \mathbf{k}_f \cdot \mathbf{J} + i\eta/2]^{-1}, \end{aligned} \quad (17a)$$

where

$$a = (v_f^2 - K^2 + 2\varepsilon_f)/2K. \quad (17b)$$

The second-order internuclear amplitude ($A_p^{(2)}$) can be further simplified by the same method described in Paper I for the second-order positron-electron amplitude $A_e^{(2)}$ as

$$\begin{aligned} A_p^{(2)} &= C_2 \int_0^\infty dx e^{iax} x^{-i(v_{pT}^a + v_{pe}^a)} \int_0^\infty dk_i k_i^{l_i+2} (k_i^2 - 2\varepsilon_{n_i})^{-l_i-2+i v_{pe}^a} C_{n_i-l_i-1}^{l_i+1} \left(\frac{k_i^2 + 2\varepsilon_{n_i}}{k_i^2 - 2\varepsilon_{n_i}} \right) j_{l_i}(k_i x) \\ &\quad \times \int_0^\infty dk_f k_f^{l_f+2} (k_f^2 - \varepsilon_{n_f})^{-l_f-2+i v_{pT}^a} C_{n_f-l_f-1}^{l_f+1} \left(\frac{k_f^2 + \varepsilon_{n_f}}{k_f^2 - \varepsilon_{n_f}} \right) j_{l_f}(bk_f x), \end{aligned} \quad (18a)$$

with

$$\begin{aligned} C_2 &= -2^{5+i(-v_{pe}^a + 2v_{pT}^a - 2v_{pT}^a)} \pi Z_p Z_T N_{n_i l_i} N_{n_f l_f} \exp \left[i \frac{\pi}{2} (l_i - l_f - 1) \right] \\ &\quad \times \frac{\exp \left[-\frac{5\pi}{2} (v_{pe}^a + v_{pT}^a) \right]}{\Gamma(1 - i v_{pe}^a - i v_{pT}^a)} \times \frac{\Gamma(1 - i v_{pT}^a)^2 \Gamma(1 + i v_{pT}^a)}{\Gamma(1 - i v_{pT}^a)} \times \frac{\Gamma(1 - i v_{pe}^a)^2 \Gamma(1 + i v_{pe}^a)}{\Gamma(1 - i v_{pe}^a)} \\ &\quad \times \sqrt{v_i^{2i(v_{pe}^a - 2v_{pe}^a + v_{pT}^a - 2v_{pT}^a)}} \times J^{-2-2i v_{pT}^a} K^{-3-2i v_{pe}^a + i v_{pT}^a + i v_{pe}^a} Y_{l_i m_i}(\hat{\mathbf{K}}) Y_{l_f m_f}^*(\hat{\mathbf{J}}) \end{aligned} \quad (18b)$$

and $b=J/K$. The expansion Eq. (12c) for the Gegenbauer polynomial is next substituted in Eq. (18a) which further simplifies it to

$$\begin{aligned} A_p^{(2)} &= C_2 \sum_{s_i=0}^{n_i-l_i-1} \sum_{s_f=0}^{n_f-l_f-1} A_{n_i l_i s_i} A_{n_f l_f s_f} \int_0^\infty dx e^{iax} x^{-i(v_{pT}^a + v_{pe}^a)} \int_0^\infty dk_i k_i^{l_i+2} (k_i^2 - 2\varepsilon_{n_i})^{-l_i-s_i-2+i v_{pe}^a} j_{l_i}(k_i x) \\ &\quad \times \int_0^\infty dk_f k_f^{l_f+2} (k_f^2 - \varepsilon_{n_f})^{-l_f-s_f-2+i v_{pT}^a} j_{l_f}(bk_f x). \end{aligned} \quad (18c)$$

The derivation now continues in a way similar to the procedure leading to the term $A_e^{(2)}$ that was discussed in Paper I. This similarity is evidence for the importance of the term $A_p^{(2)}$, as well as for the second-order electronic term, in explaining the occurrence of the Thomas peak at the angle of 45° . Both these terms have a maximum at $a=0$, where classically the condition ($a=0$) guaranteed the conservation of energy and momentum of the system during the collision. On the other hand, this equality ($a=0$) also occurs at scattering angles of about 45° which is the critical Thomas peak angle. The final result of our derivation is given by

$$A_P^{(2)} = \sum_{s_i=0}^{n_i-l_i-1} \sum_{s_f=0}^{n_f-l_f-1} P_{n_i l_i \nu_{pe}^a n_f l_f \nu_{PT}^a}(\varepsilon_{n_i}, \varepsilon_{n_f}) \mathcal{J}(1 + l_i + l_f + s_i + s_f - 2i\nu_{pe}^a - 2i\nu_{PT}^a, a; \frac{1}{2} + s_i - i\nu_{pe}^a, \frac{1}{2} + s_f - i\nu_{PT}^a; \sqrt{-2\varepsilon_{n_i}}, b\sqrt{-\varepsilon_{n_f}}), \quad (19a)$$

in which

$$P_{n_i l_i \nu_{pe}^a n_f l_f \nu_{PT}^a}(\varepsilon_{n_i}, \varepsilon_{n_f}) = \frac{2^{-(l_i+s_i-i\nu_{pe}^a+l_f+s_f-i\nu_{PT}^a+3)} \pi}{\Gamma(l_i + s_i + 2 - i\nu_{pe}^a) \Gamma(l_f + s_f + 2 - i\nu_{PT}^a)} A_{n_i l_i s_i} A_{n_f l_f s_f} \times (-2\varepsilon_{n_i})^{-(2s_i+1-2i\nu_{pe}^a)/4} (-\varepsilon_{n_f})^{-(2s_f+1-2i\nu_{PT}^a)/4} C_2, \quad (19b)$$

while Eqs. (19a) and (15) might look complicated; in practice, they are relatively simple to compute with a personal computer being more than sufficient. The A_{FWL} can then be calculated by summing up the partial amplitudes evaluated in Eq. (1) or Eq. (3b), using Eqs. (15) and (19a) for $A_T^{(2)}$ and $A_P^{(2)}$, respectively, and Eq. 51 of Paper I for $A_e^{(2)}$. The first-order terms are obtained from the results of other works.²³ Differential cross sections in the laboratory and center of mass frames are approximately equal as

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{C.M.}} = \frac{1}{\pi^2} \frac{v_f}{v_i} |A_{FWL}|^2 \quad (20)$$

and are therefore computable by using the second-order FWL amplitude from Eq. (1).

IV. DISCUSSION AND CONCLUSION

Positronium formation by the impact of a positron with an atomic target was described here under a three-body model. The FWL formalism was adopted, as it is a fully quantum mechanical approach to describe the charge transfer process. The second-order FWL formalism contains five terms, as was shown schematically in Fig. 1. In this paper closed forms for the two nuclear terms were specifically evaluated, as their importance to the scattering process was described in a review by Dewangan and Eichler.²⁴

In Fig. 3(a), the second-order internuclear amplitudes associated with the target, $A_T^{(2)}$, and the projectile, $A_P^{(2)}$, are plotted and compared with our earlier positron-electron amplitude as calculated in Paper I. The second-order internuclear term, associated with the projectile, shows the Thomas peak as expected and is roughly of the same order of magnitude as the second-order positron-electron contribution. Specifically, this nuclear term is about 25% less at the Thomas peak, while it is smaller by about 50% at angles higher than the Thomas peak. The second-order internuclear term associated with the target is an order of magnitude smaller than the other two terms below the Thomas peak and at smaller angles, while all three terms are roughly equal in magnitude at around a scattering angle of 60°. The second-order internuclear term associated with the target then dominates the other two terms at larger angles, being close to two orders of magnitude higher in the backward direction.

The phases of the three terms are now plotted and compared in Fig. 3(b). It is clear from this figure that the phase of the second-order internuclear term associated with the target is constant over all angles. This term has a constructive effect on the second-order contribution to the final cross section at higher angles, while the second-order internuclear term associated with the projectile has a destructive effect. The highest-order term affecting the cross section is the first Born term, described earlier,¹⁹ which has a phase of $(-\pi)$. Note that when a term has a phase difference of an odd multiple of π from the first Born term, it will have a destructive effect, while a phase difference of an even multiple of π will produce a constructive effect on the final cross section.

The calculated reaction amplitude was inserted into Eq. (20) in order to calculate the differential cross sections for the $1s$ to $1s$ transition in the positronium formation reaction of e^+H . Unfortunately, there are currently no experimental data available for those differential cross sec-

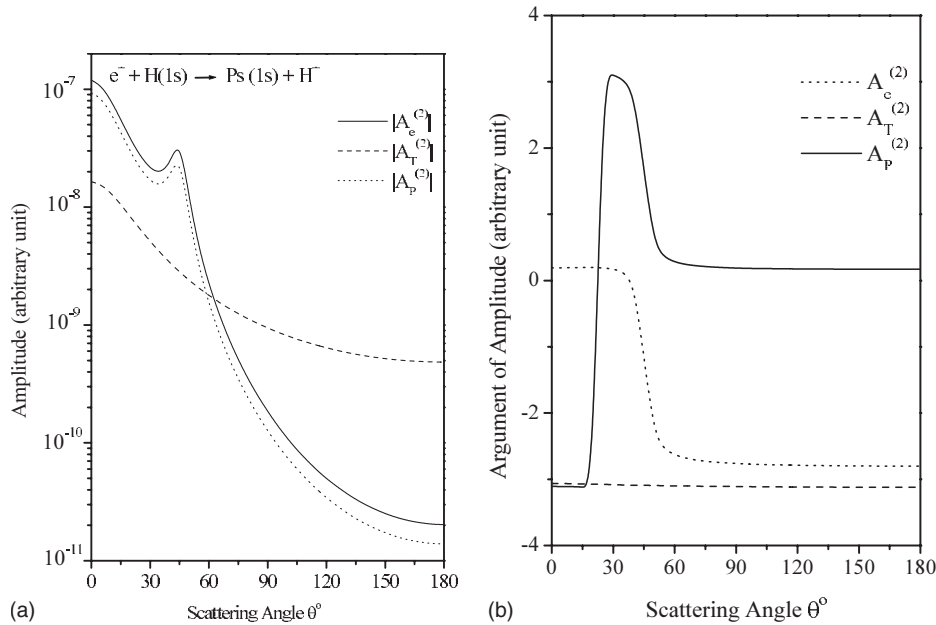


FIG. 3. The present results for the three nuclear terms in the second-order amplitude as calculated under the FWL formalism. The absolute values of the amplitudes are compared in (a) and the phases of the terms are compared in (b). The illustrated data are for 50 keV positron impact on atomic hydrogen.

tions in the literature for us to compare with. However, we have integrated our results over the scattering angles to calculate the total cross sections for the positronium reaction of e^+H . This enabled us to compare the present results with the available experimental and other theoretical total cross sections in the literature, as shown in Fig. 4. It should be noted that the range of energy for the experimental results (13–55 eV) is quite different from those of the present theoretical studies (50–50 000 eV). Further note that a line is drawn in Fig. 4 to guide the eye. However, this line has the energy dependences of E^{-6} and $E^{-5.5}$ for the lower and higher energy regions, respectively, with our calculated energy dependence for the cross sections being between (E^{-5}) and (E^{-6}) at higher energies.¹⁷

As anticipated, there is good agreement between the theories in the higher energy region, where it is expected that the present Faddeev formalism will be valid. Even at the lower energies

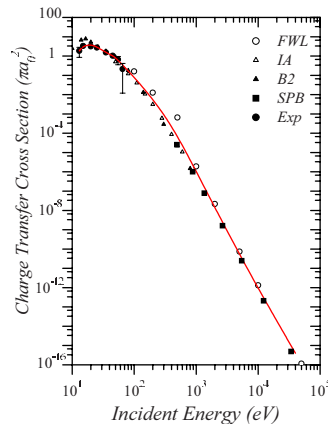


FIG. 4. (Color online) The present results for the total capture cross sections are plotted as a function of incident positron energy. The results from other experimental work (Refs. 14 and 15), and IA (Ref. 16), SPB (Ref. 17), and second Born approximation (Ref. 18) calculations are also plotted for comparison. A line is drawn to guide the eye.

the agreement is fair, tending well to the available experimental data. For future work, we will try to adopt a Faddeev–Born approach to study the charge transfer reaction for positronium formation, which is a better approach at small scattering angles. Experimental data are also still needed to better understand this apparently simple three-body reaction.

ACKNOWLEDGMENTS

This work was partially supported by the International Center for Science and High Technology and the Environmental Science, Mahan, Iran. One of us (M.J.B.) thanks the Australian Research Council for some financial support under its Center of Excellence Programme. Finally, F.S. would like to acknowledge support from the Ministry of Science, Research and Technology, Iran.

- ¹R. D. Rivarola, *Phys. Scr.* **74**, C24 (2006).
- ²D. Kleppner, *Rev. Mod. Phys.* **71**, S78 (1999).
- ³L. D. Faddeev, *Sov. Phys. JETP* **12**, 1014 (1961).
- ⁴K. M. Watson, *Phys. Rev.* **88**, 1163 (1952).
- ⁵C. Lovelace, *Phys. Rev. B* **135**, B1225 (1964).
- ⁶G. R. Newton, *Scattering Theory of Waves and Particles* (Springer, New York, 1982).
- ⁷C. J. Joachain, *Quantum Collision Theory*, 3rd ed. (North-Holland, Amsterdam, 1987).
- ⁸B. H. Bransden, *Atomic Collision Theory*, 2nd ed. (Benjamin-Cummings, Reading, MA, 1983).
- ⁹J. C. Y. Chen, *Case Stud. At. Phys.* **3**, 305 (1974).
- ¹⁰S. Alston, *Phys. Rev. A* **42**, 331 (1990); S. Alston, *ibid.* **38**, 1701 (1988).
- ¹¹M. J. Roberts, *J. Phys. B* **37**, 2869 (2004).
- ¹²D. Fischer, K. Stöckel, H. Cederquist, H. Zettergren, P. Reinhard, R. Schuch, A. Källberg, A. Simonsson, and H. T. Schmidt, *Phys. Rev. A* **73**, 052713 (2006).
- ¹³M. A. Bolorizadeh, M. J. Brunger, T. Maddern, and E. Ghanbari-Adivi, *J. Math. Phys.* **48**, 033506 (2007).
- ¹⁴W. Sperber, Ph.D. thesis, Universität Bielefeld, 1993.
- ¹⁵W. Sperber, D. Becke, K. G. Lynn, W. Raith, A. Schweb, A. Sinapius, G. Spicher, and M. Weber, *Phys. Rev. Lett.* **68**, 3690 (1992).
- ¹⁶S. J. Ward and J. H. Macek, *Hyperfine Interact.* **89**, 477 (1994).
- ¹⁷J. H. McGuire, N. C. Sil, and N. C. Deb, *Phys. Rev. A* **34**, 685 (1986).
- ¹⁸A. Igarashi and N. Toshima, *Phys. Rev. A* **47**, 2386 (1993).
- ¹⁹K. Dettmann, *Springer Tracts Mod. Phys.* **58**, 119 (1971).
- ²⁰S. Alston, *Phys. Rev. A* **38**, 636 (1988).
- ²¹S. Alston, *Nucl. Instrum. Methods Phys. Res. B* **43**, 19 (1989).
- ²²I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Academic, New York, 1980).
- ²³Authors have calculated the first-order term of $A_{PT}^{(1)}$, which is not published. Additionally, the first Born term A_{B1} is determined from Ref. 19.
- ²⁴D. P. Dewangan and J. Eichler, *Phys. Rep.* **247**, 59 (1994).